On a representation of the inverse F_q-transform

Sabir Umarov[1], Constantino Tsallis[2,3]

[1] *Department of Mathematics, Tufts University, Medford, MA 02155, USA*
[2] *Centro Brasileiro de Pesquisas Fisicas, Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil*
[3] *Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA*

Abstract

A representation formula for the inverse q-Fourier transform is obtained in the class of functions $\mathcal{G} = \bigcup_{1 \le q < 3} \mathcal{G}_q$, where $\mathcal{G}_q = \{f = a e_q^{-\beta x^2},\ a > 0,\ \beta > 0\}$.

In this paper we find a representation formula for the inverse F_q-transform in a class of functions $\mathcal{G}$ defined below. F_q-transform is a useful tool in the study of limit processes in nonextensive statistical mechanics (see [1] and references therein). Note that, in this theory, random states are correlated in a special manner, and a knowledge on F_q-inverse of data is helpful in understanding the nature of such correlations.

Throughout the paper we assume that $1 \le q < 3$. The F_q*-transform*, called also *q-Fourier transform*, of a nonnegative $f(x) \in L_1(R^1)$ is defined by the formula (see [2, 3])

$$F_q[f](\xi) = \int_{supp\, f} e_q^{ix\xi} \otimes_q f(x) dx\,, \tag{1}$$

where $\otimes_q$ is the symbol of the q-product, and e_q^x is a q-exponential. The reader is referred for details of q-algebra and q-functions to [4, 5, 6]. F_q coincides with the classic Fourier transform if $q = 1$. If $1 < q < 3$ then F_q is a nonlinear mapping in L_1. The representation formula for the inverse, F_q^{-1}, is defined in the class of function of the form $Ae_{q_1}^{-B\xi^2}$, since it uses a specific operator I defined in this class. The question on extension of this operator to wider classes is remaining a challenging question.

The obvious equality $e_q^{ix\xi} \otimes_q f(x) = f(x) e_q^{ix\xi[f(x)]^{q-1}}$, which holds for all $x \in supp\, f$, implies the following lemma, which gives an expression for the q-Fourier transform without usage of the q-product.

Proposition 0.1 *The q-Fourier transform can be written in the form*

$$F_q[f](\xi) = \int_{supp\, f} f(x) e_q^{ix\xi[f(x)]^{q-1}} dx. \tag{2}$$

Introduce the operator

$$F_q^*[f](\xi) = \int_{supp\, f} f(x) e_q^{-ix\xi[f(x)]^{q-1}} dx. \tag{3}$$

For arbitrary nonnegative $f \in L_1(R)$ both operators, F_q and F_q^* are correctly defined. Moreover,

$$\sup_{\xi \in R^1} |F_q[f](\xi)| \le \|f\|_{L_1} \ \text{ and } \ \sup_{\xi \in R^1} |F_q^*[f](\xi)| \le \|f\|_{L_1}. \tag{4}$$

Introduce the set of functions

$$\mathcal{G}_q = \{f : f(x) = a e_q^{-\beta x^2},\ a > 0,\ \beta > 0\}. \tag{5}$$

Obviously, $\mathcal{G}_q \subset L_1$ for all $q < 3$. The set $\mathcal{G}_q$ is fully identified by the triplet (q, a, β). We denote

$$\mathcal{R} = \{(q, a, \beta) : q < 3, a > 0, \beta > 0\}.$$

For any function $f \in \mathcal{G}_q$ we have $f(-x) = f(x)$, so f is symmetric about the origin. Moreover, it follows from the symmetry that $F_q[f](\xi) = F_q^*[f](\xi)$. Further, a function $f \in \mathcal{G}_q$ with $a = \frac{\sqrt{\beta}}{C_q}$, where

$$C_q = \begin{cases} \frac{2}{\sqrt{1-q}} \int_0^{\pi/2} (\cos\, t)^{\frac{3-q}{1-q}} dt = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1}{1-q}\right)}{(3-q)\sqrt{1-q}\,\Gamma\left(\frac{3-q}{2(1-q)}\right)}, & -\infty < q < 1, \\ \sqrt{\pi}, & q = 1, \\ \frac{2}{\sqrt{q-1}} \int_0^{\infty} (1+y^2)^{\frac{-1}{q-1}} dy = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\,\Gamma\left(\frac{1}{q-1}\right)}, & 1 < q < 3\,. \end{cases} \tag{6}$$

is called a q-Gaussian, and is denoted by $G_q(\beta; x)$. Thus, the set of all q-Gaussians forms a subset of $\mathcal{G}_q$.

The following statement was proved in [2].

Proposition 0.2 *Let $1 \le q < 3$. For the q-Fourier transform of a q-Gaussian, the following formula holds:*

$$F_q[G_q(\beta; x)](\xi) = \left(e_q^{-\frac{\xi^2}{4\beta^{2-q} C_q^{2(q-1)}}}\right)^{\frac{3-q}{2}}. \tag{7}$$

Assume a sequence q_k is given by

$$q_k = \frac{2q - k(q-1)}{2 - k(q-1)},\ -\infty < k < \frac{2}{q-1} - 1, \tag{8}$$

for $q > 1$, and $q_k = 1$ for all $k = 0, \pm 1, ...,$ if $q = 1$.

It follows from this proposition the following result.

Corollary 0.3 *Let $1 \le q < 3$ and $k < \frac{2}{q-1} - 1$. Then*

$$F_{q_k}[G_{q_k}(\beta; x)](\xi) = e_{q_{k+1}}^{-\beta_{k+1}\xi^2}, \tag{9}$$

where $q_{k+1} = \frac{1+q_k}{3-q_k}$ and $\beta_{k+1} = \frac{3-q_k}{8\beta^{2-q_k} C_{q_k}^{2(q_k-1)}}$.

Remark 0.4 *It follows from (9) that*

$$F_{q_k}[a e_{q_k}^{-\beta x^2}](\xi) = \frac{a C_{q_k}}{\sqrt{\beta}}\ e_{q_{k+1}}^{-B\xi^2},$$

where $B = \frac{a^{2(q_k-1)}(3-q_k)}{8\beta}$.

Theorem 0.5 *The operator $F_{q_k} : \mathcal{G}_{q_k} \to \mathcal{G}_{q_{k+1}}$ is invertible.*

Proof. With the operator $F_{q_k} : \mathcal{G}_{q_k} \to \mathcal{G}_{q_{k+1}}$ we associate the mapping $\mathcal{R} \to \mathcal{R}$ defined as $(q_k, a, \beta) \to (q_{k+1}, A, B)$, where $A = \frac{aC_{q_k}}{\sqrt{\beta}}$ and $B = \frac{a^{2(q_k-1)}(3-q_k)}{8\beta}$. Consider the system of equations

$$\frac{1+q_k}{3-q_k} = Q,$$

$$\frac{aC_{q_k}}{\sqrt{\beta}} = A,$$

$$\frac{a^{2(q_k-1)}(3-q_k)}{8\beta} = B$$

with respect to q_k, a, β assuming that Q, A and B are given. The first equation is autonomous and has a unique solution $q_k = (3Q-1)/(Q+1)$. If the condition $k < \frac{2}{q-1} - 1$ is fulfilled then the other two equations have a unique solution as well, namely

$$a = \left(\frac{A\sqrt{3-q_k}}{2C_{q_k}\sqrt{2B}} \right)^{\frac{1}{2-q_k}}, \ \beta = \left(\frac{A^{2(q_k-1)}(3-q_k)}{8C_{q_k}^{2(q_k-1)}B} \right)^{\frac{1}{2-q_k}}. \tag{10}$$

It follows from (8) that Q and q_k are related as $Q = q_{k+1}$. Hence, the inverse mapping $(F_q)^{-1} : \mathcal{G}_{k+1} \to \mathcal{G}_k$ exists and maps each element $Ae_{q_{k+1}}^{-B\xi^2} \in \mathcal{G}_{k+1}$ to the element $ae_{q_k}^{-\beta x^2} \in \mathcal{G}_k$ with a and β defined in (10). ■

Now we find a representation formula for the inverse operator F_q^{-1}. Denote by T the mapping $T : (a, \beta) \to (A, B)$, where $A = \frac{aC_{q_k}}{\sqrt{\beta}}$ and $B = \frac{a^{2(q_k-1)}(3-q_k)}{8\beta}$, as indicated above. We have seen that T is invertible and $T^{-1} : (A, B) \to (a, \beta)$ with a and b in (10). Assume $(\bar{a}, \bar{\beta}) = T^{-2}(A, B) = T^{-1}(T^{-1}(A, B)) = T^{-1}(a, \beta)$. Further, we introduce the operator $I_{(q_{k+1}, q_{k-1})} : \mathcal{G}_{q_{k+1}} \to \mathcal{G}_{q_{k-1}}$ defined by the formula

$$I_{(q_{k+1}, q_{k-1})}[Ae_{q_{k+1}}^{-B\xi^2}] = \bar{a}e_{q_{k-1}}^{-\bar{\beta}\xi^2}. \tag{11}$$

Consider the composition $H_{q_k} = F_{q_{k-1}}^* \circ I_{(q_{k+1}, q_{k-1})}$. By definition, it is clear that $I_{(q_{k+1}, q_{k-1})} : \mathcal{G}_{q_{k+1}} \to \mathcal{G}_{q_{k-1}}$. Since $F_{q_{k-1}}^* : \mathcal{G}_{q_{k-1}} \to \mathcal{G}_{q_k}$, we have $H_{q_k} : \mathcal{G}_{q_{k+1}} \to \mathcal{G}_{q_k}$. Let $\hat{f} \in \mathcal{G}_{k+1}$, that is $\hat{f}(\xi) = Ae_{q_{k+1}}^{-B\xi^2}$. Then, taking into account the fact that *supp* $\hat{f} = R^1$ if $q \geq 1$, one obtains an explicit form of the operator H_{q_k} :

$$H_{q_k}[\hat{f}(\xi)](x) = \int_{-\infty}^{\infty} \left(\bar{a}e_{q_{k-1}}^{-\bar{\beta}\xi^2} \right) \otimes_{q_{k-1}} e_{q_{k-1}}^{-ix\xi} d\xi = \int_{-\infty}^{\infty} I_{(q_{k+1}, q_{k-1})}[\hat{f}(\xi)] \otimes_{q_{k-1}} e_{q_{k-1}}^{-ix\xi} d\xi. \tag{12}$$

Theorem 0.6 *1. Let $f \in \mathcal{G}_{q_k}$. Then $H_{q_k} \circ F_{q_k}[f] = f$;*

2. Let $f \in \mathcal{G}_{q_{k+1}}$. Then $F_{q_k} \circ H_{q_k}[f] = f$.

Proof. 1. We need to show validity of the equation $F_{q_{k-1}} \circ I_{(q_{k+1}, q_{k-1})} \circ F_{q_k} = \mathcal{J}$, where $\mathcal{J}$ is the identity operator in $\mathcal{G}_{q_k}$. This equation is equivalent to $T \circ T^{-2} \circ T = J$ (J is the identity operator in $\mathcal{R}$ with fixed q), which is correct by construction.

2. Now the equation $F_{q_k} \circ F_{q_{k-1}} \circ I_{(q_{k+1}, q_{k-1})} = \mathcal{J}'$ ($\mathcal{J}'$ is the identity operator in $\mathcal{G}_{q_{k+1}}$) is equivalent to the identity $T^2 \circ T^{-2} = J$. ■

Corollary 0.7 *The operator $H_{q_k} : G_{q_{k+1}} \to G_{q_k}$ is the inverse to the q_k-Fourier transform: $H_{q_k} = F_{q_k}^{-1}$.*

Corollary 0.8 *For $q = 1$ the inverse $F_{q_k}^{-1}$ coincides with the classical inverse Fourier transform.*

Proof. If $q = 1$, then by definition one has $q_k = q_{k-1} = q_{k+1} = 1$. We find $\bar{a}$ and $\bar{\beta}$ taking $(A, B) = (1, 1)$. It follows from relationships (10) that $(a, \beta) = T^{-1}(1, 1) = (\frac{1}{2\sqrt{\pi}}, \frac{1}{4})$. Again using (10) we obtain $(\bar{a}, \bar{\beta}) = T^{-2}(1, 1) = T^{-1}(\frac{1}{2\sqrt{\pi}}, \frac{1}{4}) = (\frac{1}{2\pi}, 1)$. This means that $I_{(1,1)}\hat{f}(\xi) = \frac{1}{2\pi}\hat{f}(\xi)$. Hence, the formula (12) takes the form

$$F_1^{-1}[\hat{f}(\xi)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-ix\xi}dx,$$

recovering the classic formula for the inverse Fourier transform. ∎

Summarizing, we have proved that, if $\hat{f}(\xi)$ is a function in $\mathcal{G}_{q_{k+1}}$, where q_k with $k < \frac{2}{q-1} - 1$ is defined in Eq. (8) for $q \in [1, 3)$, then

$$F_{q_k}^{-1}[\hat{f}(\xi)](x) = \int_{-\infty}^{\infty} I_{(q_{k+1},q_{k-1})}[\hat{f}(\xi)] \otimes_{q_{k-1}} e_{q_{k-1}}^{-ix\xi} d\xi, \tag{13}$$

with the operator $I_{(q_{k+1},q_{k-1})}$ given in (11). This might constitute a first step for finding a representation of $F_q^{-1}[\hat{f}(\xi)](x)$ for generic $\hat{f}(\xi) \in L_1(R)$, which would be of great usefulness.

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